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Pseudounitary symmetry and the Gaussian pseudounitary ensemble of random matrices

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Employing the currently discussed notion of pseudo-Hermiticity, we define a pseudounitary group. Further, we develop a random matrix theory that is invariant under such a group and call this ensemble of pseudo-Hermitian random matrices the pseudounitary ensemble. We obtain exact results for the nearest-neighbor level-spacing distribution for (2×2) $\mathcal{PT}$-invariant Hamiltonian matrices that have forms, $\sim S\ln(1/S)$ near zero spacing for three independent elements and $\sim S$ for four independent elements. This shows a level repulsion in a marked distinction with an algebraic form S^{β} in the Wigner surmise. We believe that this paves the way for a description of varied phenomena in two-dimensional statistical mechanics, quantum chromodynamics, and so on.

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Postulates of quantum theory require the observables to be represented by Hermitian operators as only real eigenvalues correspond to measurements. However, it has recently been emphasized that there are certain Hamiltonians describing the quantum systems that possess real eigenvalues even though they are not Hermitian. Many of these systems are invariant under space-time reflection, i.e., invariant under a joint action of parity $\mathcal{P}$ and time reversal $\mathcal{T}$ [1–3]. In this context, the concept of pseudo-Hermiticity was introduced [4], where it was shown that $\mathcal{PT}$ symmetry is a special case of pseudo-Hermiticity. Pseudo-Hermiticity of an operator or a matrix $\mathbf{O}$ is simply defined through the condition $\mathbf{O}^{\dagger}=\boldsymbol{\eta}\mathbf{O}\boldsymbol{\eta}^{-1}$ with $\boldsymbol{\eta}$ as a metric and "†" representing the usual adjoint or conjugate transpose. Although a pseudo-Hermitian Hamiltonian may also possess complex-conjugate pairs as eigenvalues, we restrict our discussion to the case where the spectrum is real. Remarkably, it was subsequently shown that non-$\mathcal{PT}$ invariant systems that possess real eigenvalues are also pseudo-Hermitian [5]. Physical situations of great interest belong to the above discussion. This includes two-dimensional statistical mechanics where parity and time-reversal are broken (preserving $\mathcal{PT}$) [6–8], quantum chromodynamics where chiral ensembles are used to describe the statistical properties of lattice Dirac operator [9], spin-rotation coupling leading to an anomalous g value for muon [10], and related fields. In this paper, we present a random matrix theory that describes spectral fluctuations in systems that are pseudo-Hermitian and pseudounitarily invariant. The two aspects that are particularly notable are the simplicity of this description and the fact that this theory is natural when parity or (and) time reversal is (are) violated.

The problem of two-dimensional statistical mechanics is obviously connected with anyon physics and hence to the behavior of an electron in an Aharonov-Bohm medium [11], i.e., a medium filled with nonquantized magnetic fluxes, reminiscent of the theory of fractional quantum Hall effect [12]. Important to note here is also another motivation that stems from a speculation due to Nambu that this might serve as a model for theoretical ideas such as the quark confinement in a medium of monopoles [13]. In this context, it is known that the spectral fluctuations of an Aharonov-Bohm billiard exhibits an interpolating behavior with respect to the strength of the flux line [14]. These billiards are experimentally realized in terms of quantum dots in the presence of flux lines. It is of great interest to find an appropriate random matrix description for such $\mathcal{PT}$-invariant systems. Pseudo-Hermiticity appears in several contexts. It is instructive to note that in the mean-field random phase approximation description of nuclei [15], the stability matrix leading to an eigenvalue problem can be checked to be pseudounitary. In context of the regularization of quantum field theories, pseudo-Hermiticity and the associated improper metric was used by Dirac [16], Pauli [17], and particularly by Gupta and Bleuler [18], and others [19]. Let us first establish the pseudounitary symmetry.

Consider vectors $\mathbf{x}$ and $\mathbf{y}$ residing in a vector space $\mathcal{V}$ and a fixed metric $\boldsymbol{\eta}$. In this vector space, we define a pseudo–inner product ($\boldsymbol{\eta}$ norm), which can be written in the usual quantum mechanical notation as $\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{y}\rangle$. We shall consider symmetry transformations that preserve the $\boldsymbol{\eta}$ norm between the vectors. We consider the Cayley form $\mathbf{D}=e^{i\mathbf{G}}$ as a symmetry transformation acting on $\mathbf{x}$, $\mathbf{y}$, where $\mathbf{G}$ is pseudo-Hermitian in accordance with $\boldsymbol{\eta}\mathbf{G}\boldsymbol{\eta}^{-1}=\mathbf{G}^{\dagger}$. By noting an interesting feature of $\mathbf{D}$:

$$\mathbf{D}^{\dagger}=e^{-i\mathbf{G}^{\dagger}}=e^{-i\boldsymbol{\eta}\mathbf{G}\boldsymbol{\eta}^{-1}}=\boldsymbol{\eta}e^{-i\mathbf{G}}\boldsymbol{\eta}^{-1}=\boldsymbol{\eta}\mathbf{D}^{-1}\boldsymbol{\eta}^{-1}, \quad (1)$$

let us call $\mathbf{D}$ as pseudounitary with respect to $\boldsymbol{\eta}$. $\boldsymbol{\eta}$ equal to unity makes $\mathbf{D}$ unitary trivially. To establish that $\mathbf{D}$ is indeed a symmetry transformation, we need to show that the transformation preserves the $\boldsymbol{\eta}$ norm and a consistently defined matrix element.

Let us assume that $\mathbf{x}(\mathbf{y})\rightarrow\mathbf{x}'(\mathbf{y}')=\mathbf{D}\mathbf{x}(\mathbf{D}\mathbf{y})$. Then, the pseudounitary symmetry is defined by preserving the pseudonorm

$$\langle \mathbf{x}'|\boldsymbol{\eta}\mathbf{y}'\rangle=\langle \mathbf{D}\mathbf{x}|\boldsymbol{\eta}\mathbf{D}\mathbf{y}\rangle=\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{y}\rangle. \quad (2)$$

In proving Eq. (2), use $e^{-iG^{\dagger}}\boldsymbol{\eta}e^{iG}=e^{-iG^{\dagger}}\boldsymbol{\eta}e^{iG}\boldsymbol{\eta}^{-1}\boldsymbol{\eta}=e^{-iG^{\dagger}}e^{iG^{\dagger}}\boldsymbol{\eta}=\boldsymbol{\eta}$. Under the same pseudounitary transformation, the matrix element of an arbitrary operator $\mathbf{A}$ transforms as

$$\langle \mathbf{x}'|\boldsymbol{\eta}\mathbf{A}'|\mathbf{y}'\rangle=\langle \mathbf{x}|\boldsymbol{\eta}\mathbf{A}|\mathbf{y}\rangle \quad \text{if } \mathbf{D}\mathbf{A}\mathbf{D}^{-1}=\mathbf{A}'. \quad (3)$$

Let us now prove that pseudounitary matrices form a group under matrix multiplication. For closure, let $\mathbf{D}_1$ and $\mathbf{D}_2$ be two pseudounitary matrices. $\mathbf{D}_1\mathbf{D}_2$ is pseudounitary because $\boldsymbol{\eta}^{-1}(\mathbf{D}_1\mathbf{D}_2)^\dagger\boldsymbol{\eta}=\boldsymbol{\eta}^{-1}\mathbf{D}_2^\dagger\boldsymbol{\eta}\boldsymbol{\eta}^{-1}\mathbf{D}_1^\dagger\boldsymbol{\eta}=(\mathbf{D}_1\mathbf{D}_2)^{-1}$. It easily follows that $\mathbf{D}^{-1}$ is pseudounitary with respect to $\boldsymbol{\eta}$ if $\mathbf{D}$ is pseudounitary: $\boldsymbol{\eta}^{-1}(e^{-i\mathbf{G}})^\dagger\boldsymbol{\eta}=e^{i\boldsymbol{\eta}^{-1}\mathbf{G}^\dagger\boldsymbol{\eta}}=e^{i\mathbf{G}}$. The identity matrix acts as the unit element of the symmetry transformation. Finally, since the associativity is guaranteed, the $N\times N$ pseudounitary matrices form a pseudounitary group of order N, PU(N).

In the following, to keep the proceedings simple and explicit, we consider Hamiltonians in their matrix representations. Also, in the spirit of the original work of Wigner [20], we consider (2×2) matrices as they bring out most of the essence. In this context, there is a recent generalization of Wigner surmise for 2×2 matrices [21]. Thus, we concentrate on PU(2) and consider the following pseudo-Hermitian matrix:

$$\mathbf{H}=\{\mathbf{H}_{ij}\}=\begin{bmatrix} a & -ib \\ ic & a \end{bmatrix}, \tag{4}$$

a,b,c being real. Consequently, $e^{i\mathbf{H}}$ will be a pseudounitary matrix. For the above matrices, a metric is

$$\boldsymbol{\delta}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{5}$$

This metric may be interpreted as the parity operator $\mathcal{P}$, and the complex conjugation $\mathcal{K}_0$ as time-reversal operator $\mathcal{T}$. With these operations, it may be verified that $\mathbf{H}$ is $\mathcal{PT}$ invariant in addition to being $\mathcal{P}$-pseudo-Hermitian. Besides these commuting $\mathcal{P}$ and $\mathcal{T}$ operators, if we choose $\mathcal{T}$ as the Pauli matrix times the complex conjugation, $\sigma_x\mathcal{K}_0$, they do not commute, preserving other conclusions.

We want to emphasize that $\boldsymbol{\delta}$ need not be unique. For instance, it could be σ_y, diagonal matrices diag$(c/b,1)$, diag$(\sqrt{c/b},\ \sqrt{b/c})$, ... Further, we insist that the metric should be independent of the matrix elements and we call such a metric "secular." This also disentangles the metric with probability distribution of the matrix elements (13), in anticipation with the subsequent discussion on random matrices.

This group admits three generators and an identity, viz.,

$$\boldsymbol{\rho}_1=\begin{bmatrix} 1 & 0 \\ i & -1 \end{bmatrix},\quad \boldsymbol{\rho}_2=\begin{bmatrix} 1 & -i \\ 0 & -1 \end{bmatrix},$$
$$\boldsymbol{\rho}_3=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\quad \mathbf{I}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

Note that $\mathbf{H}=a\mathbf{I}+c\boldsymbol{\rho}_1+b\boldsymbol{\rho}_2+(b+c)\boldsymbol{\rho}_3$. It is interesting to see that $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are pseudo-Hermitian and pseudounitary, possessing eigenvalues ± 1. It may be recalled that the Pauli matrices σ_x and σ_y are Hermitian and unitary. Further, the generators satisfy the following important properties:

$$\boldsymbol{\rho}_1^2=\boldsymbol{\rho}_2^2=\boldsymbol{\rho}_3^2=\mathbf{I},$$

$$[\boldsymbol{\rho}_i,\boldsymbol{\rho}_j]=\sum_k C_{ij}^k\boldsymbol{\rho}_k, \tag{7}$$

with $C_{12}^1=C_{12}^2=C_{23}^2=C_{23}^3=C_{31}^1=C_{31}^3=2$ and $C_{12}^3=5$. All the structure constants can be found with the help of commutation relations and symmetry properties, and they turn out to be ± 5, ± 2, or 0. Interestingly, the following relations between the structure constants hold:

$$C_{kl}^j=-C_{lk}^j,$$
$$\sum_{m=1}^{3}[C_{kl}^m C_{jm}^s+C_{lj}^m C_{km}^s+C_{jk}^m C_{lm}^s]=0, \tag{8}$$

thus making it a Lie group and defining a Lie algebra [23].

We now consider a Hamiltonian $\mathbf{H}$, which is diagonalizable by $\mathbf{D}$, i.e.,

$$\mathbf{H}=\mathbf{D}\begin{bmatrix} E_+ & 0 \\ 0 & E_- \end{bmatrix}\mathbf{D}^{-1}. \tag{9}$$

The eigenvalues of $\mathbf{H}$ are $a\pm\sqrt{bc}$ (b and c are of the same sign). The corresponding matrix, $\mathbf{D}$,

$$\mathbf{D}=\begin{bmatrix} 1 & i/r \\ ir & 1 \end{bmatrix}, \tag{10}$$

is pseudounitary under the metric

$$\boldsymbol{\eta}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

Had the secular $\boldsymbol{\delta}$ been diagonal, $\boldsymbol{\eta}$ would have been same as $\boldsymbol{\delta}$. The matrix $\mathbf{D}$ generates an ensemble of pseudo-Hermitian matrices through Eq. (9) which have the general form (4) involving three independent parameters. The most general form of $\mathbf{H}$ would, of course, involve four independent parameters. We return to this point after illustrating the nature of spectral fluctuations for Eq. (4).

The eigenvalues are

$$E_\pm=a\pm\left[\frac{c}{2r}+\frac{br}{2}\right], \tag{12}$$

where $r=\sqrt{c/b}\,(0\leqslant r\leqslant\infty)$.

Consider that the matrix $\mathbf{H}$ is drawn from an ensemble of random matrices with a Gaussian distribution given by [20]

$$P(\mathbf{H})=\mathcal{N}e^{-(1/2\sigma^2)\mathrm{tr}(\mathbf{H}^\dagger\mathbf{H})}. \tag{13}$$

Accordingly, the joint probability distribution of a,b,c is

$$P(a,b,c)=\frac{1}{2(\pi\sigma^2)^{3/2}}e^{-(1/2\sigma^2)[2a^2+b^2+c^2]}. \tag{14}$$

From Eqs. (4) and (9), we have the following relations:

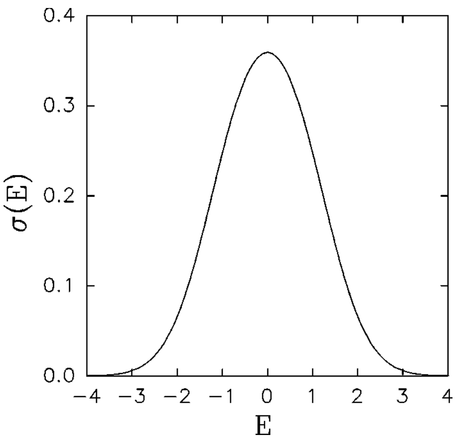

FIG. 1. The average level density of an ensemble of 2×2 random Gaussian pseudounitary ensembles is shown here.

$$a=\frac{E_+ + E_-}{2},\quad b=\frac{E_+ - E_-}{2r},\quad c=\frac{r(E_+ - E_-)}{2}. \tag{15}$$

The Jacobian J connecting (a,b,c) and (E_+,E_-,r) is $|E_+ - E_-|/2r$. With these, the joint probability distribution function (JPDF) of eigenvalues is

$$P(E_+,E_-)=\frac{|E_+ - E_-|}{2(\pi\sigma^2)^{3/2}}K_0\left(\frac{(E_+ - E_-)^2}{4\sigma^2}\right)e^{-(E_+ + E_-)^2/4\sigma^2}. \tag{16}$$

Following the Dyson-Coulomb gas analogy, this JPDF can be written as an equilibrium distribution of two interacting particles with a partition function $P(E_+,E_-)\rightarrow\mathcal{Z}(x_1,x_2)=e^{-\beta\mathcal{H}(x_1,x_2)}$. It is interesting to note that $\mathcal{H}$ has a potential term involving the logarithm of the modified Bessel function along with the familiar harmonic confinement and the two-dimensional Coulomb potential. $4\sigma^2$ plays the role of inverse scaled temperature.

Integrating with respect to E_- gives the average density, shown in Fig. 1. This is not amenable to an analytically closed form.

Perhaps the most well-studied characterizer is the nearest-neighbor level spacing distribution, $P(S)$. This gives the frequency with which a certain spacing between adjacent levels occurs. For the Wigner-Dyson ensembles, $P(S)\sim S^{\beta_0}e^{-\gamma S^2}$, where β_0 is 1, 2, and 4 for the orthogonal, unitary, and symplectic ensembles, respectively. A wide variety of systems display universal properties possessed by random matrix ensembles as can be seen in Refs. [20,24,25]. However, there are systems that display intermediate statistics [26–28]. These systems range from examples of billiards in polygonal enclosures, three-dimensional Anderson model at the metal-insulator transition point, and so on. On the other hand, there have been important developments on non-Hermitian ensembles since long where the eigenvalues are complex [20,25], and where an ensemble of unstable states is considered [29]. Clearly, the ensemble developed here does not fall into any of the known categories and, indeed, displays some different features as shown below.

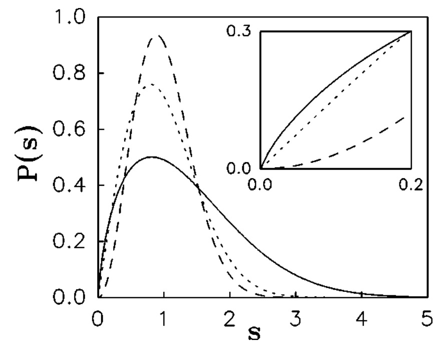

FIG. 2. The nearest-neighbor level-spacing distribution is shown here. For comparison, the results corresponding to the Wigner-Dyson ensembles corresponding to orthogonal and unitary symmetries are also shown, whereas the level repulsion is linear and quadratic in the orthogonal and unitary ensembles, here it is of the form $S\ln(1/S)$, as shown in the inset. This then suggests a different universality.

The spacing distribution $P(S)$ is given in terms of the JPDF by

$$P(S)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P(E_+,E_-)\,\delta(S-|E_+ - E_-|)dE_+dE_- = \frac{S}{\pi\sigma^2}K_0\left(\frac{S^2}{4\sigma^2}\right). \tag{17}$$

This result is distinctly different and very interesting (Fig. 2), particularly for its behavior near zero spacing. Near $S=0$, the probability distribution varies as $S\ln1/S$. This follows from the asymptotic properties of the modified Bessel function.

The question now is to establish the generality of Eq. (17). Having made the metrics (5) and (11) secular, the independence of Eq. (17) from the metric(s) trivially follows. We find that [30] for other forms of three-parameter pseudo-Hermitian Hamiltonian matrices **H**, viz.,

$$\begin{bmatrix} a+ib & c \\ c & a-ib \end{bmatrix},\quad \begin{bmatrix} a+c & ib \\ ib & a-c \end{bmatrix},$$

$$\begin{bmatrix} a+ib & -ic \\ ic & a-ib \end{bmatrix}\quad (c^2>b^2),\quad \begin{bmatrix} a & \pm b \\ c & a \end{bmatrix}\quad (\mp bc<0), \tag{18}$$

we get Eq. (17) for $P(S)$ with distribution (13).

The root of this generality lies in the mathematical form of the level spacing which is real only conditionally in contrast with the absolute reality for the Wigner-Dyson ensembles [20]. That this aspect is basically due to the pseudo-Hermitian character of the matrices is worth keeping in mind.

We now return to the case of pseudo-Hermitian Hamiltonian matrices with four independent parameters and real eigenvalues. There are two general cases which are exhaustive, viz., where the diagonal elements are complex conjugate and where they are real. Once again, the level spacing will be real only conditionally. We find that [30] the spacing distribution behaves as $\sim S\,\exp(S^2)\mathrm{erfc}(S)$ near zero spacing, where $\mathrm{erfc}(x)$ is the complementary error function. The degree of level repulsion is linear here. Most importantly, whenever these four-parameter Hamiltonians reduce to three-parameter cases, the result (17) is recovered for all the cases when pseudonorm is indefinite. We would like to remark that the spacing distribution $\sim S^2 e^{-S^2}$ for any four-parameter Hermitian matrix reduces to $\sim S e^{-S^2}$ for the Hermitian matrix with three parameters, a form that is very similar (but not identical) to that for real symmetric matrices.

It is well known that when a quantum system violates time-reversal invariance, the degree of level repulsion is two. In addition, if parity is broken, the degree of level repulsion becomes one, as if it is a restoration of time-reversal invariance. This scenario is when there are four statistically independent parameters. Further, when there remain three independent parameters, the level repulsion becomes nonalgebraic $[S \ln 1/S]$.

Finally, we point out an aspect of general importance encountered on many occasions in many-body theory. To give one concrete example, in the theory of collective excitations of fermionic systems, a mean-field description is used where a collective state is first expressed in terms of particle-hole excitations [15]. Here, one generally encounters a matrix equation such as $\mathbf{H}\Psi = \lambda\Phi$, with $\mathbf{H}$ a Hermitian or unitary operator. The above problem may be transformed into an eigenvalue problem for $\mathbf{H}'$, i.e., $\mathbf{H}'\Phi = \lambda\Phi$, with $\mathbf{H}'$ a pseudo-Hermitian or pseudounitary operator. With this, there are many results immediately possible. First of all, the eigenvalues will either be real, complex-conjugate pairs, unimodular, or they occur in pairs such that the product of eigenvalues is unimodular [22]. Second, the statistical properties of the eigenvalues related to collective excitations will be distributed in accordance with the results obtained for the Gaussian pseudounitary ensemble (GPUE) above.

The above results are found for 2×2 pseudo-Hermitian matrices. For $N\times N$ matrices, invariant under PU(N), where not much is known, we conjecture that the fluctuation properties may have a similar form as above, taking cue from the Wigner surmise for $N\times N$ matrices. As discussed earlier, the general results found here may suggest a different universality corresponding to systems that are pseudounitarily invariant. In such systems, parity and time reversal may be individually broken, preserving $\mathcal{PT}$. This universality might also include those pseudo-Hermitian quantum systems where $\mathcal{PT}$ is broken. The examples discussed include quantum chromodynamics, two-dimensional statistical mechanics, and so on.

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